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PERIODIC MOTIONS OF GYROSCOPIC SYSTEMS*

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A generalized conservative gyroscopic system is considered. It is shown that there is a two-parameter family of periodic solutions of the complete equations of motion of the system, close to the similar family of solutions of the precession equations.

1. Consider a conservative mechanical system which contains l gyroscopes. We assume that the system position is defined by $2m+l$ generalized coordinates $x_1, \ldots, x_{2m}, \varphi_1, \ldots, \varphi_l$, where $\varphi_1, \ldots, \varphi_l$ are the angles of proper rotation of the gyroscopes, while $x=(x_1, \ldots, x_{2m})^T$ are parameters which characterize the directions of the gyroscope axes and the positions of the suspensions. We also assume that the system is described by the Lagrange function /1/

$$L=\frac{1}{2}\sum_{i,j=1}^{2m} a_{ij}(x)x_i^{\cdot}x_j^{\cdot}+\frac{1}{2}\sum_{k=1}^{l} C_k\left(\varphi_k^{\cdot}+\sum_{i=1}^{2m} a_i^{(k)}(x)x_i^{\cdot}\right)^2-\Pi(x)$$

Here, the dot denotes differentiation with respect to time t, C_k are constants, and the symmetric matrix $A(x)=(a_{ij}(x))_{i,j=1}^{2m}$ is positive definite. The angles φ_k are cyclical coordinates, and the corresponding first integrals are

$$\frac{\partial L}{\partial \varphi_k^{\cdot}}=C_k\left(\varphi_k^{\cdot}+\sum_{i=1}^{2m} a_i^{(k)}(x)x_i^{\cdot}\right)=h_k \quad (k=1, \ldots, l)$$

Using Rouse's method and introducing the notation

$$hg_{ij}(x)=\sum_{k=1}^{l} h_k\left(\frac{\partial a_i^{(k)}}{\partial x_j}-\frac{\partial a_j^{(k)}}{\partial x_i}\right) \quad (i, j=1, \ldots, 2m)$$

the equations of motion of the system can be written as

$$\sum_{j=1}^{2m} a_{ij}x_j^{\cdot\cdot}+\sum_{j,k=1}^{2m}\left(\frac{\partial a_{ij}}{\partial x_k}-\frac{1}{2}\frac{\partial a_{jk}}{\partial x_i}\right)x_j^{\cdot}x_k^{\cdot}= -h\sum_{j=1}^{2m} g_{ij}x_j^{\cdot}-\frac{\partial \Pi}{\partial x_i} \quad (i=1, \ldots, 2m) \tag{1.1}$$

These equations have the generalized energy integral

$$\frac{1}{2}\sum_{i,j=1}^{2m} a_{ij}(x)x_i^{\cdot}x_j^{\cdot}+\Pi(x)=\text{const} \tag{1.2}$$

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We shall assume that the h in Eqs.(1.1) is a large positive parameter, and that the functions $\Pi(x)$, $a_{ij}(x)$, $g_{ij}(x)$ $(i, j = 1, \ldots, 2m)$ are independent of h. We shall also assume that the matrix $G(x) = (g_{ij}(x))_{i,j=1}^{2m}$ is not degenerate. The equations

$$h \sum_{j=1}^{2m} g_{ij} x_j^{\cdot} + \frac{\partial \Pi}{\partial x_i} = 0 \quad (i = 1, \ldots, 2m) \tag{1.3}$$

which have the first integral $\Pi(x) = \text{const}$, are called precessional /1/. We shall assume that these equations have a two-parameter family of periodic solutions, and we shall examine whether the complete Eqs.(1.1) have a similar family.

To give an exact statement of the problem, we change in (1.1) and (1.3) to the new independent variable $\tau = h^{-1}t$. To simplify the writing we shall use vector notation. We then obtain the equations

$$Gx' + \left(\frac{\partial \Pi}{\partial x}\right)^T = -h^{-2}\left(\frac{d}{d\tau}\frac{\partial R}{\partial x'} - \frac{\partial R}{\partial x}\right)^T, \quad R = \frac{1}{2}(x')^T A(x) x' \tag{1.4}$$

$$Gx' + \left(\frac{\partial \Pi}{\partial x}\right)^T = 0 \tag{1.5}$$

The prime here denotes differentiation with respect to τ, and the derivatives of the scalar functions with respect to vector arguments are regarded as row vectors, e.g., $\partial\Pi/\partial x = (\partial\Pi/\partial x_1, \ldots, \partial\Pi/\partial x_{2m})$. We assume that $A(x)$, $G(x)$, and $\Pi(x)$ are fairly smooth functions, i.e., have all the derivatives required for our future working.

System (1.5) is obtained from (1.4) with $h = \infty$, so that we shall call the system degenerate. We shall assume the following with regard to (1.5):

1°. The system admits of the two-parameter family of periodic solutions

$$x = \varphi(\tau + \tau_0, c) \tag{1.6}$$

with period $T = T(c)$, where $c \in (c_1^\circ, c_2^\circ)$, $\tau_0 \in (-\infty, +\infty)$ are parameters.

2°. With $c \in [c_1, c_2] \subset (c_1^\circ, c_2^\circ)$, the system of equations in variations for the solution (1.6) has a non-trivial T-periodic solution $\varphi'(\tau + \tau_0, c)$ which is unique, apart from a constant factor.

3°. With $c \in [c_1, c_2]$ we have $\partial\Pi(\varphi(\tau + \tau_0, c))/\partial x \not\equiv 0$.

Without loss of generality, we can put $\tau_0 = 0$ in (1.6). We shall seek the T-periodic solutions $x(\tau, c, h)$ of system (1.4) which are defined for values (c, h) of an unbounded set $I_h \subset [c_1, c_2] \times [0, +\infty)$ and satisfy with admissible $h \to \infty$ the conditions $x(\tau, c, h) \to \varphi(\tau, c)$, $x'(\tau, c, h) \to \varphi'(\tau, c)$.

2. We first make some auxiliary transformations. Since the matrix $A(x)$ is symmetric and positive definite, while $G(x)$ is skew-symmetric, the corresponding bilinear forms can be simultaneously reduced to canonical form /2/. More precisely, there is a non-degenerate matrix $F(x)$ such that

$$\begin{gathered} F^T(x)A(x)F(x) = E_{2m} \\ F^T(x)G(x)F(x) = -\operatorname{diag}(\gamma_1(x)J, \ldots, \gamma_m(x)J) \equiv -\Gamma(x) \\ J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \end{gathered} \tag{2.1}$$

Here and below, E_k is the k-th order unit matrix.

The conditions 1°-3° of Sect.1 are supplemented by conditions 4°.

For all admissible x the matrix $F(x)$ and the scalars $\gamma_j(x)$ $(j = 1, \ldots, m)$, are fairly smooth functions.

5°. For $c \in [c_1, c_2]$, $\tau \in (-\infty, +\infty)$, we have

$$0 < \gamma_1^2[\varphi(\tau, c)] < \gamma_2^2[\varphi, (\tau, c)] < \ldots < \gamma_m^2[\varphi(\tau, c)]$$

To reduce system (1.4) to normal form, we introduce the new variable $p \in R^{2m}$, putting

$$x' = F(x)p + \Phi(x), \quad \Phi(x) = -G^{-1}(x)(\partial\Pi(x)/\partial x)^T \tag{2.2}$$

We substitute our expression for x' into (1.4) and multiply the result by $F^T(x)$. Using (2.1), we obtain the equation

$$p' = h^2\Gamma(x)p + f(x, p) \tag{2.3}$$

where $f(x, p)$ is a second degree polynomial in p with coefficients which depend on $A(x)$, $F(x)$, $\Phi(x)$ and their first derivatives. Eqs.(2.2) and (2.3) form a closed system, equivalent to system (1.4).

In system (2.2), (2.3) we make the change variable $x = \varphi(\tau, c) + \xi$ and we isolate some terms explicitly in the resulting equations. We thus arrive at the T-periodic system

$$\begin{aligned}
&\xi' = A(\tau, c)\xi + B(\tau, c)p + \Phi_1(\tau, \xi, p, c) \qquad (2.4)\\
&p' = \{h^2[\Gamma_0(\tau, c) + \Gamma_1(\tau, \xi, c)] + C(\tau, c)\}p + f_0(\tau, c) + \\
&\quad f_1(\tau, \xi, p, c) + h^2 g_1(\tau, \xi, p, c)\\
&A(\tau, c) = \frac{\partial\Phi(\varphi)}{\partial x}, \quad B(\tau, c) = F(\varphi), \quad C(\tau, c) = \frac{\partial f(\varphi, 0)}{\partial p}\\
&f_0(\tau, c) = f(\varphi, 0), \quad \Gamma_0(\tau, c) = \Gamma(\varphi)\\
&\Gamma_1(\tau, \xi, c) = \operatorname{diag}\left[\frac{\partial\gamma_1(\varphi)}{\partial x}\xi J, \ldots, \frac{\partial\gamma_m(\varphi)}{\partial x}\xi J\right], \quad \varphi = \varphi(\tau, c)
\end{aligned}$$

in which we have, for the functions Φ_1, f_1, and g_1, uniformly with respect to $c \in [c_1, c_2]$, $\tau \in (-\infty, +\infty)$, the estimates

$$\begin{aligned}
&\Phi_1(\tau, \xi, p, c) = O(\|\xi\|^2 + \|\xi\|\|p\|)\\
&f_1(\tau, \xi, p, c) = O(\|\xi\| + \|p\|^2), \quad g_1(\tau, \xi, p, c) = O(\|\xi\|^2\|p\|)
\end{aligned}$$

Here, $\|\cdot\|$ is the Euclidean norm.

In the new variables, we can write the first integral (1.2) as

$$\begin{aligned}
&\Pi(\varphi + \xi) + h^{-2}\Pi_1(\tau, \xi, p, c) = \text{const} \qquad (2.5)\\
&(\Pi_1(\tau + T, \xi, p, c) = \Pi_1(\tau, \xi, p, c))
\end{aligned}$$

Since (2.5) is the first integral of system (2.4), its total derivative with respect to τ is, by virtue of this system, identically zero. Putting $\xi = p = 0$ in this identity and its partial derivative with respect to p, and separating in the resulting relations the principal terms as $h \to \infty$, we find the equations required below:

$$\begin{aligned}
&\frac{\partial\Pi_1(\tau, 0, 0, c)}{\partial\tau} + \frac{\partial\Pi_1(\tau, 0, 0, c)}{\partial p} f_0(\tau, c) = 0 \qquad (2.6)\\
&\frac{\partial\Pi(\varphi)}{\partial x} B(\tau, c) + \frac{\partial\Pi_1(\tau, 0, 0, c)}{\partial p}\Gamma_0(\tau, c) = 0
\end{aligned}$$

Our subsequent transformations are used to simplify the linear terms in system (2.4). The substitution $\xi = u + h^{-2}B(\tau, c)\Gamma_0^{-1}(\tau, c)p$ reduces this system to

$$\begin{aligned}
&u' = A(\tau, c)u + h^{-2}\Phi_0(\tau, c) - \Phi_2(X) \qquad (2.7)\\
&p' = \{h^2[\Gamma_0(\tau, c) + \Gamma_1(\tau, u, c)] + C(\tau, c)\}p + f_0(\tau, c) + \\
&\quad f_2(X) + h^2 g_2(X)\\
&\Phi_0(\tau, c) = -B(\tau, c)\Gamma_0^{-1}(\tau, c)f_0(\tau, c), \quad X = (\tau, u, p, c, h)
\end{aligned}$$

where, for the functions Φ_2, f_2, and g_2 as $u, p, h^{-1} \to 0$ uniformly with respect to $c \in [c_1, c_2]$ and $\tau \in (-\infty, +\infty)$, we have the estimates

$$\begin{aligned}
&\Phi_2(X) = O[h^{-2}(\|u\| + \|p\|) + \|u\|\|p\| + \|u\|^2]\\
&f_2(X) = O(\|u\| + h^{-2}\|p\| + \|p\|^2), \quad g_2(X) = O(\|u\|^2\|p\|)
\end{aligned}$$

Consider the linear inhomogeneous system

$$u' = A(\tau, c)u + \Phi_0(\tau, c) \qquad (2.8)$$

We know /3/ that this system has a T-periodic solution if and only if, given any T-periodic solution $\psi(\tau)$ of the adjoint system

$$\psi' + \psi A(\tau, c) = 0, \quad \psi^T \in R^{2m} \qquad (2.9)$$

we have the equation

$$\int_0^T \psi(\tau)\Phi_0(\tau, c)\,d\tau = 0$$

Since, by condition 2° of Sect.1, the system

$$u' = A(\tau, c)u \qquad (2.10)$$

has a unique non-trivial T-periodic solution $u = \varphi'(\tau, c)$, then system (2.9) likewise has a unique non-trivial T-periodic solution /3/. In accordance with /4/ and condition 3°, this solution can be written as

$$\psi = \partial \Pi (\varphi)/\partial x \equiv \psi_0 (\tau, c) \tag{2.11}$$

Using (2.6) and the definition of $\Phi_0 (\tau, c)$, we find that

$$\int_0^T \psi_0 (\tau, c)\, \Phi_0 (\tau, c)\, d\tau = -\int_0^T \frac{\partial \Pi (\varphi)}{\partial x} B \Gamma_0^{-1} f_0\, d\tau =$$
$$\int_0^T \frac{\partial \Pi_1 (\tau, 0, 0, c)}{\partial p} f_0\, d\tau = -\int_0^T \frac{\partial \Pi_1 (\tau, 0, 0, c)}{\partial \tau} d\tau = 0$$

Consequently, a T-periodic solution of system (2.8) exists. Call it $u_0 (\tau, c)$. It is defined apart from a term proportional to $\varphi' (\tau, c)$. To fix $u_0 (\tau, c)$, we require satisfaction of the condition

$$\int_0^T [\varphi' (\tau, c)]^T u_0 (\tau, c)\, d\tau = 0$$

A method of constructing the solution $u_0 (\tau, c)$ is given below. We merely mention here that it is a smooth function of c.

In system (2.7) we make the replacement

$$u = y + h^{-2} u_0 (\tau, c) \tag{2.12}$$

We obtain

$$\begin{gathered} y' = A (\tau, c) y + \Phi_3 (Y) \\ p' = [h^2 \Gamma_0 (\tau, c) + C_1 (\tau, c)] p + f_3 (Y) + h^2 g_3 (Y) \\ C_1 (\tau, c) = C (\tau, c) + \Gamma_1 (\tau, u_0 (\tau, c), c), \quad Y = (\tau, y, p, c, h) \end{gathered} \tag{2.13}$$

As $y, p, h^{-1} \to 0$ uniformly with respect to $c \in [c_1, c_2]$, $\tau \in (-\infty, +\infty)$, the functions Φ_3, f_3, and g_3 satisfy the estimates

$$\begin{gathered} \Phi_3^\circ (\tau, c, h) = O (h^{-4}), \quad f_3^\circ (\tau, c, h) = O (1), \quad f_3^{\circ\prime} (\tau, c, h) = O(1) \\ \Phi_3 (Y) - \Phi_3^\circ (\tau, c, h) = O [h^{-2} (\| y \| + \| p \|) + \| y \|^2 + \\ \| y \| \| p \|] \\ f_3 (Y) - f_3^\circ (\tau, c, h) = O (\| y \| + h^{-2} \| p \| + \| p \|^2) \\ g_3 (Y) = O (\| p \| \| y \|) \end{gathered} \tag{2.14}$$

In these relations and below, for any function $g (\tau, \cdot, \cdot, c, h)$ we use the notation $g^\circ (\tau, c, h) = g (\tau, 0, 0, c, h)$.

The transformation (2.12) reduces the right-hand side of the first equation of the system. The next transformation simplifies the term $C_1 (\tau, c) p$ in the second of Eqs. (2.13). Instead of p, we introduce the variable

$$z = [E_{2m} + h^{-2} Q (\tau, c)] p$$

where $Q (\tau, c)$ is a T-periodic matrix. The explicit form of $Q (\tau, c)$ is given below. As a result, system (2.13) transforms to

$$\begin{gathered} y' = A (\tau, c) y + \Phi_4 (U) \\ z' = [h^2 \Gamma_0 (\tau, c) + D (\tau, c)] z + f_4 (U) + h^2 g_4 (U) \\ D (\tau, c) = C_1 + Q \Gamma_0 - \Gamma_0 Q, \quad U = (\tau, y, z, c, h) \end{gathered} \tag{2.15}$$

The matrices Q and D are obtained as follows. We write matrices C_1, Q, and D in the 2×2 block form: $C_1 = (C_{ij})_{i,j=1}{}^m$, $D = (D_{ij})_{i,j=1}{}^m$, $Q = (Q_{ij})_{i,j=1}{}^m$. We can then write the expressions for Q as the system

$$D_{ij} = C_{ij} + \gamma_j^\circ Q_{ij} J - \gamma_i^\circ J Q_{ij}, \quad \gamma_i^\circ = \gamma_i (\varphi) \quad (i, j = 1, \ldots, m)$$

An indirect check in the light of condition 5° of Sect.2 shows that we can take

$$\begin{gathered} Q_{ij} = (\gamma_j^{\circ 2} - \gamma_i^{\circ 2})^{-1} (\gamma_i^\circ J C_{ij} + \gamma_j^\circ C_{ij} J), \quad D_{ij} = 0 \quad (i \neq j) \\ Q_{ii} = (4 \gamma_i^\circ)^{-1} (C_{ii} J - J C_{ii}), \; D_{ii} = {}^1/_2 (C_{ii} - J C_{ii} J) = \\ \mu_i E_2 + \nu_i J \\ \mu_i = \mu_i (\tau, c) = {}^1/_2 \operatorname{tr} C_{ii}, \; \nu_i = \nu_i (\tau, c) = -{}^1/_2 \operatorname{tr} J C_{ii} \end{gathered}$$

The matrices $Q(\tau, c)$ and $D(\tau, c)$ thus constructed are T-periodic with respect to τ. As $y, z, h^{-1} \to 0$ uniformly with respect to $c \in [c_1, c_2]$, $\tau \in (-\infty, +\infty)$, the functions Φ_4, f_4 and g_4 in (2.15) satisfy estimates similar to (2.14).

We introduce the unbounded set

$$I(\varepsilon) = \{(c, h)\colon c_1 \leqslant c \leqslant c_2,\ h \geqslant 0,\ \Delta_i(c, h) \geqslant \varepsilon\ (i = 1, \ldots, m)\}$$

$$\varepsilon \in (0, 1), \quad \Delta_i(c, h) = \operatorname{sh}^2 a_i(c) + \sin^2 [h^2 b_i(c) + d_i(c)]$$

$$a_i(c) = \frac{1}{2} \int_0^T \mu_i(\tau, c)\, d\tau, \quad b_i(c) = \frac{1}{2} \int_0^T \gamma_i^\circ(\tau, c)\, d\tau,$$

$$d_i(c) = \frac{1}{2} \int_0^T \nu_i(\tau, c)\, d\tau$$

Theorem. Given any $\varepsilon \in (0, 1)$, positive numbers H, A_1, and A_2 exist such that, for $(c, h) \in I(\varepsilon)$, $h \geqslant H$, system (2.15) has a unique $T(c)$-periodic solution $y_*(\tau, c, h)$, $z_*(\tau, c, h)$ which satisfies the conditions

$$\| y_*(\tau, c, h)\| \leqslant A_1 h^{-4}, \quad \| z_*(\tau, c, h)\| \leqslant A_2 h^{-2} \tag{2.16}$$

$$(-\infty < \tau < +\infty), \quad \int_0^T [\varphi'(\tau, c)]^T y_*(\tau, c, h)\, d\tau = 0$$

Notes. 1°. The last condition of (2.16) is used to fix a time shift which is permissible in the solutions of autonomous systems.

2°. The condition $(c, h) \in I(\varepsilon)$ eliminates from the analysis of periodic solutions of system (2.15) resonances between the slow oscillations (with frequency $2\pi/T$) and the fast oscillations (with frequencies $\sim h^2$). Such resonances can arise at values of c and h given by the equations $a_i(c) = 0$, $\sin[h^2 b_i(c) + d_i(c)] = 0$, where i is any of the numbers $1, \ldots, m$. To each root c_* of the first equation there corresponds a set of resonance values h: $h_n = \sqrt{(\pi n - d_i(c_*))/b_i(c_*)}$. Here, n is any integer for which the expression under the root is positive. Large h and $|n|$ are of interest. In this case,

$$h_n = \sqrt{\frac{\pi n}{b_i(c_*)}} - \frac{d_i(c_*)}{2\sqrt{\pi n b_i(c_*)}} + O(|n|^{-3/2})$$

It can be seen from this relation that we only need to know the functions $d_i(c)$ when calculating the second term in the asymptotic form of the resonance values h; and this term approaches zero as $h \to \infty$. On the other hand, the determination of $d_i(c)$ is the most difficult part of the working required for our theorem. For, in order to find $b_i(c)$ we have to know $\gamma_i^\circ(\tau, c)$, and to find $a_i(c)$ we have to know the diagonal elements of the matrix $C(\tau, c)$ (which are the same as the diagonal elements of $C_1(\tau, c)$), while to find $d_i(c)$ we need to find the two collateral diagonals of $C_1(\tau, c)$, the evaluation of which demands that we find the periodic solution of Eq.(2.8). By what has been said, when applying our theorem to a specific mechanical system, it may not be possible to calculate the $d_i(c)$.

3°. Corresponding to the periodic solution of system (2.15) in the theorem we have the T-periodic solution $x^\circ(\tau, c, h)$, $p^\circ(\tau, c, h)$ of system (2.2), (2.3), which satisfies the conditions

$$\| x^\circ(\tau, c, h) - \varphi(\tau, c)\| < A_1^\circ h^{-2}, \quad \| p^\circ(\tau, c, h)\| \leqslant A_2^\circ h^{-2}$$

$$(-\infty < \tau < +\infty), \quad \int_0^T [\varphi'(\tau, c)]^T x^\circ(\tau, c, h)\, d\tau = 0$$

Here, A_1° and A_2° are positive numbers. The statement of the theorem about periodic solutions in terms of system (2.2), (2.3), or of system (1.4), would contain many at first sight unjustified conditions and would be even more unwieldy, so that the statement in terms of the transformed system (2.15).

4°. It has been assumed that system (1.5) has a family of oscillatory periodic solutions i.e., that $\varphi(\tau + T(c), c) = \varphi(\tau, c)$ in (1.6). The theorem also holds in the case of a family of rotatory periodic solutions. In fact, when non-zero number T_0 and a constant vector $e \in R^{2m}$ with integral components exist such that, in (1.4)-(1.6), $A(x + T_0 e) = A(x)$, $G(x + T_0 e) = G(x)$, $\Pi(x + T_0 e) = \Pi(x)$, $\varphi(\tau + T(c), c) = \varphi(\tau, c) + T_0 e$ for all x, c, τ.

3. The proof of the theorem is based on the methods used in /4, 5/, and is very similar to the proofs of /6, 7/. We first give some auxiliary relations. We put

$$\Psi_4(U) = f_4(U) + h^2 g_4(U) \tag{3.1}$$

It was mentioned above that, for the functions Φ_4, f_4, and g_4 in system (2.15), as $y, z, h^{-1} \to 0$ uniformly with respect to $c \in [c_1, c_2]$, $\tau \in (-\infty, +\infty)$, estimates hold which are obtained from (2.14) by the replacements $3 \to 4$, $p \to z$. In view of this, there are positive numbers K, δ, H_1 such that, for all $\tau, c, h, y, z, y_1, z_1$ $(y_1, z_1 \in R^{2m})$, which satisfy the conditions $\tau \in (-\infty, +\infty)$, $c \in [c_1, c_2]$, $h \geqslant H_1$ and $\max(\|y\|, \|z\|, \|y_1\|, \|z_1\|) \leqslant \delta$, we have

$$\|\Phi_4^\circ(\tau, c, h)\| \leqslant Kh^{-4}, \quad \|\Psi_4^\circ(\tau, c, h)\| \leqslant K, \tag{3.2}$$

$$\|\Psi_4^{\circ\prime}(\tau, c, h)\| \leqslant K$$

$$\|\Phi_4(U) - \Phi_4^\circ(\tau, c, h)\| \leqslant K[h^{-2}(\|y\| + \|z\|) + \|y\|^2 + \|y\|\|z\|] \tag{3.3}$$

$$\|\Psi_4(U) - \Psi_4^\circ(\tau, c, h)\| \leqslant K(\|y\| + h^{-2}\|z\| + \|z\|^2 + h^2\|y\|\|z\|)$$

$$\|\Phi_4(U) - \Phi_4(U_1)\| \leqslant K(r_y + r_z + h^{-2})(d_y + d_z) \tag{3.4}$$

$$\|\Psi_4(U) - \Psi_4(U_1)\| \leqslant K[d_y(1 + h^2 r_z) + d_z(h^{-2} + h^2 r_y + r_z)]$$

$$U_1 = (\tau, y_1, z_1, c, h), \quad d_y = \|y - y_1\|$$

$$d_z = \|z - z_1\|, \quad r_y = \|y\| + \|y_1\|, \quad r_z = \|z\| + \|z_1\|$$

As a result of the transformations described in Sect.2, the first integral of (2.5) transforms into the first integral of system (2.15), which can be written as

$$V(U) \equiv \psi_0(\tau, c)\, y + V_1(U) = \text{const} \tag{3.5}$$

The function $\psi_0(\tau, c)$ is given by (2.11); as $y, z, h^{-1} \to 0$ uniformly with respect to $c \in [c_1, c_2]$, $\tau \in (-\infty, +\infty)$, we have the estimate $\partial V_1(U)/\partial y = O(\|y\| + h^{-2})$. It can be assumed without loss of generality that, for $c \in [c_1, c_2]$, $\tau \in (-\infty, +\infty)$, $h \geqslant H_1$ and $\max(\|y\|, \|z\|) \leqslant \delta$, we have

$$\|\partial V_1(U)/\partial y\| \leqslant K(\|y\| + h^{-2}) \tag{3.6}$$

Consider the boundary value problem

$$y(0) = y(T) \tag{3.7}$$

for the linear inhomogeneous system (see (2.8))

$$y' = A(\tau, c)\, y + \Phi(\tau) \tag{3.8}$$

corresponding to the first equation of (2.15).

Since the corresponding homogeneous system (2.10) has a non-trivial T-periodic solution $u = \varphi'(\tau, c)$, it is best to study problem (3.7), (3.8) with the aid of the generalized Green's function /4/. Denote it by $G_0(\tau, s, c)$ $(0 \leqslant \tau, s \leqslant T,\ c_1 \leqslant c \leqslant c_2)$. It is uniquely defined by the same boundary conditions and the jump condition at $\tau = s$ as the ordinary Green's function, and by the equations

$$\partial G_0(\tau, s, c)/\partial\tau = A(\tau, c)\, G_0(\tau, s, c) - n\psi_0^T(\tau, c)\,\psi_0(s, c)$$

$$\int_0^T [\varphi'(\tau, c)]^T G_0(\tau, s, c)\, d\tau = 0, \quad n = \left(\int_0^T \psi_0(\tau, c)\,\psi_0^T(\tau, c)\, d\tau\right)^{-1}$$

The expression

$$y(\tau) = \int_0^T G_0(\tau, s, c)\,\Phi(s)\, ds \tag{3.9}$$

satisfies the boundary conditions (3.7) and the relations

$$y' = A(\tau, c)\, y + \Phi(\tau) - w\psi_0^T(\tau, c)$$

$$w = n\int_0^T \psi_0(\tau, c)\,\Phi(\tau)\, d\tau, \quad \int_0^T [\varphi'(\tau, c)]^T y(\tau)\, d\tau = 0$$

It is the solution of the boundary value problem (3.7), (3.8) if and only if $w = 0$. Using (3.9), we can write explicitly the function $u_0(\tau, c)$, used in the change of variables (2.12).

We define the norm of the vector function $f(\tau)$, continuous in the interval $0 \leqslant \tau \leqslant T$, as the number

$$\nu(f) = \max_{0 \leqslant \tau \leqslant T} \| f(\tau) \|$$

For the norm of (3.9) with any $c \in [c_1, c_2]$, we have the bound

$$\nu(y) \leqslant N_0 \nu(\Phi) \tag{3.10}$$

where N_0 is a positive constant.

We now consider the boundary value problem

$$z(0) = z(T) \tag{3.11}$$

for the linear system

$$z' = [h^2 \Gamma_0(\tau, c) + D(\tau, c)] z + \Psi(\tau) \tag{3.12}$$

corresponding to the second equation of (2.15).

If $\Delta_i(c, h) > 0$ $(i = 1, \ldots, m)$, this problem has a unique solution, which, with the aid of the corresponding Green's function $G(\tau, s, c, h)$, can be written as

$$z(\tau) = \int_0^T G(\tau, s, c, h) \Psi(s)\, ds \tag{3.13}$$

In view of the special form of matrices Γ_0 and D, Eq.(3.12) is integrable in quadratures, and an explicit expression can be found for $G(\tau, s, c, h)$. We have the relations

$$G(\tau, s, c, h) = \operatorname{diag}(G_1(\tau, s, c, h), \ldots, G_m(\tau, s, c, h))$$
$$G_i(\tau, s, c, h) = G_i^\circ(\tau, s, c, h)/\Delta_i(c, h) \quad (i = 1, \ldots, m)$$

where the elements of the (2×2) matrices $G_i^\circ(\tau, s, c, h)$ are bounded piecewise smooth functions.

If $\Psi(\tau)$ in (3.12) has a continuous derivative and satisfies the boundary conditions (3.11), we find, on making the change of variable $z = v - h^{-2}\Gamma_0^{-1}(\tau, c)\Psi(\tau)$ in problem (3.11), (3.12), and applying (3.13) to the transformed problem, that

$$z(\tau) = -h^{-2}\Gamma_0^{-1}(\tau, c)\Psi(\tau) + h^{-2}\int_0^T G(\tau, s, c, h)[\Gamma_0^{-1}(s, c)\Psi'(s) +$$
$$(\Gamma_0^{-1}(s, c))'\Psi(s) - D(s, c)\Gamma_0^{-1}(s, c)\Psi(s)]\, ds$$

By the last relation and (3.13), there exist positive numbers N and H_2 such that, for $(c, h) \in I(\varepsilon)$, $h \geqslant H_2$, we have for the norm of the solution of problem (3.11), (3.12):

$$\nu(z) \leqslant \varepsilon^{-1} N \nu(\Psi) \tag{3.14}$$

$$\nu(z) \leqslant h^{-2} N [\nu(\Psi) + \varepsilon^{-1}(\nu(\Psi) + \nu(\Psi'))] \tag{3.15}$$

It is assumed below that $(c, h) \in I(\varepsilon)$, $h \geqslant \max(H_1, H_2)$, and inequalities (3.14), (3.15) are used without extra restrictions on the choice of h.

The determination of the T-periodic solutions of system (2.15) is equivalent to solving boundary value problem (3.7), (3.11) for this system. To solve the latter problem, we consider the system of integral equations

$$y(\tau) = \int_0^T G_0(\tau, s)\Phi_4(s, y(s), z(s))\, ds \equiv L_1(y, z) \tag{3.16}$$
$$z(\tau) = \int_0^T G(\tau, s)\Psi_4(s, y(s), z(s))\, ds \equiv L_2(y, z)$$

Here and below, to reduce the writing we do not usually indicate c and h among the arguments of the functions considered. We shall solve (3.16) by successive approximations. In the interval $0 \leqslant \tau \leqslant T$, we construct the sequences of functions $y_k(\tau)$, $z_k(\tau)$ $(k = 0, 1, \ldots)$, by putting

$$y_0(\tau) \equiv z_0(\tau) \equiv 0, \quad y_{k+1} = L_1(y_k, z_k) \tag{3.17}$$
$$z_{k+1} = L_2(y_k, z_k) \quad (k = 0, 1, \ldots)$$

Let us show that, given sufficiently large h, these sequences converge to the solution of system (3.16).

It can be shown that, for sufficiently large h,

$$\nu(y_k) \leqslant A_1 h^{-4} \leqslant \delta, \quad \nu(z_k) \leqslant A_2 h^{-2} \leqslant \delta \quad (k = 0, 1, \ldots) \tag{3.18}$$

where A_1 and A_2 are positive numbers.

Since

$$y_1(\tau)=\int_0^T G_0(\tau,s)\,\Phi_4^\circ(s)\,ds,\quad z_1(\tau)=\int_0^T G(\tau,s)\,\Psi_4^\circ(s)\,ds \tag{3.19}$$

we can write (3.17) with $k\geqslant 1$ as

$$y_{k+1}(\tau)=y_1(\tau)+\int_0^T G_0(\tau,s)\,[\Phi_4(s,y_k(s),z_k(s))-\Phi_4^\circ(s)]\,ds$$

$$z_{k+1}(\tau)=z_1(\tau)+\int_0^T G(\tau,s)\,[\Psi_4(s,y_k(s),z_k(s))-\Psi_4^\circ(s)]\,ds$$

We assume that $\nu(y_k)\leqslant\delta$, $\nu(z_k)\leqslant\delta$ $(k=0,1,\ldots)$. Then, by the inequalities (3.3), (3.10) and (3.14), we have

$$\begin{aligned}\nu(y_{k+1})&\leqslant\nu(y_1)+KN_0\,[h^{-2}(\nu(y_k)+\nu(z_k))+\nu^2(y_k)+\nu(y_k)\,\nu(z_k)]\\ \nu(z_{k+1})&\leqslant\nu(z_1)+\varepsilon^{-1}KN\,[\nu(y_k)+h^{-2}\nu(z_k)+\nu^2(z_k)+h^2\nu(y_k)\,\nu(z_k)]\end{aligned} \tag{3.20}$$

Applying inequalities (3.10), (3.15) to Eqs.(3.19) and using the estimates (3.2), we obtain

$$\nu(y_1)\leqslant B_1h^{-4},\quad \nu(z_1)\leqslant B_2h^{-2},\quad B_1=KN_0,\quad B_2=KN\,(1+2\varepsilon^{-1})$$

We choose A_1 and A_2 from the conditions $A_2>B_2$, $A_1>B_1+KN_0A_2$, and take

$$h\geqslant H_3=\max\left\{1,H_1,H_2,\sqrt[4]{\frac{A_1}{\delta}},\sqrt{\frac{A_2}{\delta}},\right.$$
$$\left.\sqrt{\frac{KN\,(1+A_2)\,(A_1+A_2)}{\varepsilon\,(A_2-B_2)}},\quad\sqrt{\frac{KN_0A_1\,(1+A_1+A_2)}{A_1-B_1-KN_0A_2}}\right\}$$

Then, if inequalities (3.18) hold for some k, we have by (3.20):

$$\nu(y_{k+1})\leqslant\frac{B_1+KN_0A_2}{h^4}+\frac{KN_0A_1\,(1+A_1+A_2)}{h^6}\leqslant\frac{A_1}{h^4}\leqslant\delta$$

$$\nu(z_{k+1})\leqslant\frac{B_2}{h^2}+\frac{KN\,(1+A_2)\,(A_1+A_2)}{\varepsilon h^4}\leqslant\frac{A_2}{h^2}\leqslant\delta$$

Since (3.18) hold for $k=1$, it now follows that they hold for all k.

Let us prove the convergence of iterations (3.17). Consider the sequences $\alpha_k=\nu(y_k-y_{k-1})$, $\beta_k=\nu(z_k-z_{k-1})$, $q_k=\nu(y_k)+\nu(y_{k-1})$, $r_k=\nu(z_k)+\nu(z_{k-1})$ $(k=1,2,\ldots)$. By (3.4), (3.10), and (3.14), we have

$$\alpha_{k+1}\leqslant KN_0\,(q_k+r_k+h^{-2})\,(\alpha_k+\beta_k)$$
$$\beta_{k+1}\leqslant\varepsilon^{-1}KN\,[\alpha_k\,(1+h^2r_k)+\beta_k\,(h^{-2}+h^2q_k+r_k)]$$

Assuming that $h\geqslant H_3$ and finding bounds for q_k and r_k in the last relations by means of (3.18), we obtain

$$\begin{aligned}&\alpha_{k+1}\leqslant h^{-2}M\,(\alpha_k+\beta_k),\quad \beta_{k+1}\leqslant M\,(\alpha_k+h^{-2}\beta_k)\\ &M=K\,(1+2A_1+2A_2)\max\,(N_0,\varepsilon^{-1}N)\end{aligned} \tag{3.21}$$

Consider the sequence $\rho_k=\alpha_k+h^{-1}\beta_k$ $(k=1,2,\ldots)$. By inequalities (3.21), $\rho_{k+1}\leqslant M\,(h^{-1}+h^{-2})\,\rho_k$. We introduce the set

$$I_*=\{(c,h):\,(c,h)\in I(\varepsilon),\,h\geqslant H_4\},\quad H_4=\max\,(H_3,2M+1)$$

With $(c,h)\in I_*$, we have $\rho_{k+1}\leqslant{}^1/_2\rho_k$ $(k=1,2,\ldots)$. Using this bound, we can show that the sequences $y_k(\tau)$ and $z_k(\tau)$ are uniformly convergent in the set $[0,T]\times I_*$ to continuous functions $y_*(\tau,c,h)$ and $z_*(\tau,c,h)$ which satisfy the inequalities

$$\nu(y_*)\leqslant A_1h^{-4}\leqslant\delta,\quad \nu(z_*)\leqslant A_2h^{-2}\leqslant\delta \tag{3.22}$$

Passing to the limit as $k\to\infty$ in (3.17), we find that y_*, z_* is the solution of system (3.16), these functions being continuously differentiable with respect to τ. The uniqueness of the solution is proved by the standard method.

The functions y_* and z_* satisfy the boundary conditions (3.7), (3.11), the second of Eqs.(2.15), and the equations

$$y_*'=A(\tau)\,y_*+\Phi_4(\tau,y_*,z_*)-w_*\psi_0^T(\tau) \tag{3.23}$$
$$w_*=n\int_0^T\psi_0(\tau)\,\Phi_4(\tau,y_*(\tau),z_*(\tau))\,d\tau,\quad \int_0^T[\varphi'(\tau)]^T\,y_*(\tau)\,d\tau=0$$

Let us show that $w_* = 0$, which means that y_*, z_* is the solution of boundary value problem (2.15), (3.7), (3.11). We use the device of /4/. We consider the function (see (3.5)) $V_*(\tau) = V(\tau, y_*(\tau), z_*(\tau))$. It satisfies the relations

$$0 = V_*(T) - V(0) = \int_0^T V_*'(\tau)\,d\tau = \int_0^T \Big\{ \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial y}(Ay + \Phi_4) + \tag{3.24}$$
$$\frac{\partial V}{\partial z}[(h^2\Gamma_0 + D)z + \Psi_4]\Big\}\Big|_{y=y_*,\, z=z_*} d\tau -$$
$$w_* \int_0^T \frac{\partial V}{\partial y}\Big|_{y=y_*,\, z=z_*} \psi_0^T(\tau)\,d\tau$$

Since $V(\tau, y, z)$ is the first integral of system (2.15), the expression in the braces in (3.24) is identically zero. By (3.5),

$$\partial V(\tau, y_*(\tau), z_*(\tau))/\partial y = \psi_0(\tau) + \partial V_1(\tau, y_*(\tau), z_*(\tau))/\partial y$$

where, by (3.6) and (3.22), with $(c, h) \in I_*$,

$$\nu\,[\partial V_1(\tau, y_*(\tau), z_*(\tau))/\partial y] \leqslant K(1 + A_1)h^{-2}$$

Consequently, there exists $H \geqslant H_4$ such that, with $(c, h) \in I(\varepsilon)$ and $h \geqslant H$, the last integral in (3.24) is non-zero. For these values of c and h, we have $w_* = 0$. On continuing the functions y_* and z_* T-periodically into the interval $-\infty < \tau < +\infty$, we obtain the required periodic solution of system (2.15). The theorem is proved.

4. The passage to the limit as $h \to \infty$ in Eqs.(1.1) implies an unbounded increase in the kinetic momenta of the gyroscopes when the system has finite inertial characteristics. Now consider the situation when the kinetic momenta of the gyroscopes and the generalized forces acting on the system remain finite, while the inertial characteristics tend to zero. For this, we make the replacement $\Pi(x) \to h\Pi(x)$ in (1.1), where h is a large parameter. As a result, we arrive at the equations which are obtained from (1.4) as $\tau \to t$, $h^2 \to h$. All our above constructions hold for the new equations. The only difference is a change in the time scale. However, this change of scale enables a new problem to be considered.

In (1.1) we replace the generalized potential forces $-\partial\Pi/\partial x_i$ by the generalized forces $hQ_i(t, x)$ $(i = 1, \ldots, 2m)$, which are periodic in time with period $T > 0$. Here, T is a number. Introducing the vector $Q(t, x) = (Q_1(t, x), \ldots, Q_{2m}(t, x))^T$, the new equations can be written as

$$G(x)x^{\cdot} - Q(t, x) = -h^{-1}\Big(\frac{d}{dt}\frac{\partial R}{\partial x^{\cdot}} - \frac{\partial R}{\partial x}\Big)^T, \quad R = \frac{1}{2}(x^{\cdot})^T A(x)x^{\cdot} \tag{4.1}$$

We make the same assumptions about the matrices $A(x)$ and $G(x)$ as we made in Sects.1 and 2. We also assume that the degenerate system

$$G(x)x^{\cdot} = Q(t, x)$$

has a T-periodic solution $x = \varphi(t)$ with multipliers different from 1, and that the functions $\gamma_i(x)$ in (2.1) satisfy for all t the inequalities $0 < \gamma_1^2(\varphi(t)) < \gamma_2^2(\varphi(t)) < \ldots < \gamma_m^2(\varphi(t))$. We can then prove the existence of a T-periodic solution $x(t, h)$ of system (4.1), which is defined for values of h from an unbounded set $I_h \subset [0, +\infty)$ and which, as $h \to \infty$ and $h \in I_h$, satisfies the conditions $x(t, h) \to \varphi(t)$, $x^{\cdot}(t, h) \to \varphi^{\cdot}(t)$.

The proof is almost an exact repetition of the proof given in Sects.1-3. The difference (and simplification) is as follows. First, the parameter c does not occur in the new problem, so that the set I_h is constructed on the line h, and not in the (c, h) plane. Second, the boundary value problem (3.7), (3.8) is now always solvable. Hence G_0 in system (3.16) is the ordinary Green's function, and the fact that the solution of this system is the solution of the boundary value problem (2.15), (3.7), (3.11) is trivial.

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ALGEBRAIC OPERATIONS COMPATIBLE WITH THE DYNAMICS OF A NON-LINEAR DISCRETE CONTROL SYSTEM*

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A new approach is developed to the analysis and synthesis of non-linear discrete control systems

$$x[k+1] = f(k, x[k], u[k]), \quad x \in R^n, \quad u \in R^m \tag{0.1}$$

first proposed in /1/ for continuous non-linear systems. The underlying idea of the approach is to redefine the addition of state and control vectors and multiplication of vectors by scalars in such a way that the system becomes linear in the new linear space. As an application, a description is given of a class of non-linear control systems which are isomorphic to their linear approximations, and explicit formulae for this isomorphism are presented. This makes it possible to construct a control with prescribed dynamic characteristics for the linear approximation system, using the well-developed theory of the linear case; this control is then converted via the isomorphism into a control for the non-linear system, generating the required closed-loop dynamics of the system, by introducing linear feedback that compensates for the non-linearity of the open-loop system.

1. The equation for the compatibility of the addition law in $R^m \times R^n$ with the system dynamics. We seek a composition law $\oplus_x{}^k$ on the set R^n in the form of a mapping $R^{2n} \to R^n$:

$$x'' = x \oplus_x{}^k x' \stackrel{\text{def}}{=} \varphi(k, x, x') \tag{1.1}$$

Here k is a parameter and x, x' the independent variables.

Similarly, $\oplus_u{}^k$: $R^{2m} \to R^m$:

$$u'' = u \oplus_u{}^k u' \stackrel{\text{def}}{=} \psi(k, x, x', u, u') \tag{1.2}$$

Here k, x, x' are parameters and u, u' the independent variables.

Finally, $\oplus^k$: $R^{2(m+n)} \to R^{m+n}$:

$$(u, x) \oplus^k (u', x') \stackrel{\text{def}}{=} (u \oplus_u{}^k u', x \oplus_x{}^k x'), \ (u, x) \in R^{m+n} \tag{1.3}$$

Whenever there is no need to specify k we omit the superscript k of $\oplus^k$.

Let W denote the set of all pairs of functions $u(k)$, $x(k)$, $a < k < b$, satisfying (0.1). Infinite end-points $a = -\infty$, $b = \infty$ are admissible. For any $(u[k], x[k]), (u'[k], x'[k]) \in W$, we define

$$(u[k], x[k]) \oplus (u'[k], x'[k]) \stackrel{\text{def}}{=} (u[k] \oplus_u u'[k], x[k] \oplus_x x'[k]) \tag{1.4}$$

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